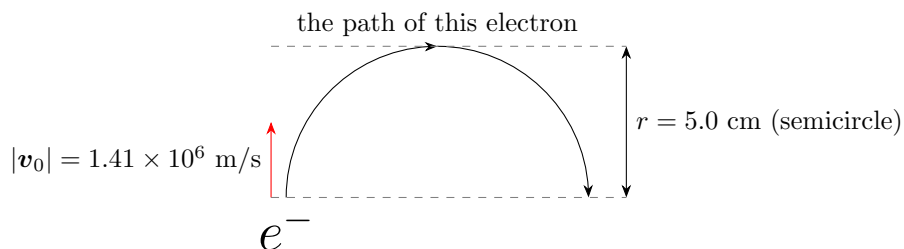


Physics 1C Discussion (Midterm 1 review)

Problem 1



Consider a motion of an electron with mass $m_e = 9.1 \times 10^{-31} \text{ kg}$ and charge $e = -1.6 \times 10^{-19} \text{ C}$. Suppose initially this electron had a speed $|v_0| = 1.41 \times 10^6 \text{ m/s}$. The electron moved under the semicircular path of radius $r = 5.0 \text{ cm}$ with a certain uniform magnetic field $\mathbf{B}$ as shown in the figure above.

(a) What is the direction of $\mathbf{B}$ for the electron to follow this semi-circular path?

Under this motion, it is satisfied that $\mathbf{F}_{\text{magnetic}} = \mathbf{F}_{\text{centripetal}}$, and

$$\mathbf{F}_{\text{magnetic}} = e\mathbf{v} \times \mathbf{B}. \quad (1)$$

Then, using the right-hand rule for the vector cross product, it is shown that the magnetic field is directed into this page (*i.e.* $\otimes$).

(b) What is the magnitude of $\mathbf{B}$ for the electron to follow this semi-circular path?

We again use the relation $\mathbf{F}_{\text{magnetic}} = \mathbf{F}_{\text{centripetal}}$. Then we get

$$\begin{aligned} |\mathbf{F}_{\text{magnetic}}| &= |e| |\mathbf{v}| |\mathbf{B}| \\ &= |e| |v_0| |\mathbf{B}| \\ &= m_e \frac{|v_0|^2}{r}. \end{aligned} \quad (2)$$

Therefore

$$\begin{aligned} |\mathbf{B}| &= \frac{m_e |v_0|}{|e| r} \\ &= \frac{(9.1 \times 10^{-31} \text{ kg}) (1.41 \times 10^6 \text{ m/s})}{(1.6 \times 10^{-19} \text{ C}) (0.05 \text{ m})} \\ &\simeq 1.6 \times 10^{-4} \text{ T} \end{aligned} \quad (3)$$

Problem 2

Suppose you want to accelerate an electron (with charge $q = -e < 0$) counter-clockwise in a circular collider with radius R . The particle starts from rest and accelerates with a constant acceleration, a .

(a) What must $\vec{B}(t)$ be to maintain a circular orbit in this accelerator?

Solution: Using the right hand rule, $B(t)$ must point out of the page ($\hat{z}$). Then,

$$q\mathbf{v}(t) \times \mathbf{B}(t) = -e\mathbf{v}(t) \times \mathbf{B}(t) = m\frac{v(t)^2}{R}(-\hat{r}) \quad (4)$$

Since the resulting direction is $\hat{r}$, using the right hand rule, B must be coming out of the page to give the correct signs. So, $|\vec{v}(t) \times \vec{B}(t)| = v(t)B(t)$. Solving for $B(t)$,

$$\mathbf{B}(t) = \frac{mv(t)}{eR}\hat{z} = \frac{mat}{eR}\hat{z} \quad (5)$$

(b) Suppose this magnetic field is created by a large circular wire with radius R' under the accelerator. Since $B(t)$ is a function of time, there will be an induced emf caused by the changing magnetic field. Calculate this emf. What direction should the current go to make the electron move counter-clockwise?

Solution:

$$\mathcal{E}_{\text{ind}} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A} = -\left(\frac{d}{dt}B(t)\right) \int dA = -\frac{dB(t)}{dt}\pi R'^2 = -\frac{ma}{eR}\pi R'^2. \quad (6)$$

Since $\hat{B} = \hat{z}$, by the right hand rule, the current must go counter-clockwise.

(c) It will turn out that charged particles (like electrons) moving around in a circle will lose some energy due to heat. The formula for the energy loss per turn is given by,

$$E_{\text{loss}}^p = \frac{q_p^2}{3\epsilon_0 m_p^4} \frac{E^4}{R}, \quad (7)$$

where E is the energy of the particle. To minimize the energy loss, some physicists have considered using muons ($m_\mu = 1.88 \times 10^{-28}$, $q_\mu = -1.60 \times 10^{-19}$) instead of electrons, ($m_e = 9.11 \times 10^{-31}$, $q_e = -1.60 \times 10^{-19}$). To see one reason why, calculate $E_{\text{loss}}^\mu/E_{\text{loss}}^e$, assuming the energy of the particles is the same. What do your results tell you? Given your solutions to the previous parts, can you think of a downside of using muons?

Solution:

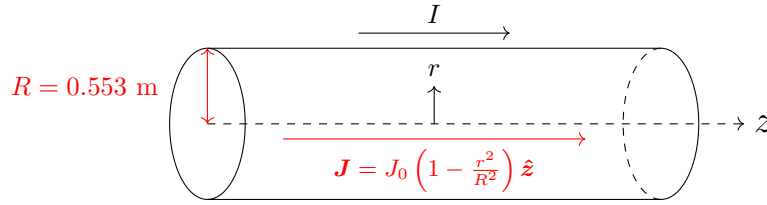
$$\frac{E_{\text{loss}}^\mu}{E_{\text{loss}}^e} = \frac{q_\mu^2}{3\epsilon_0 m_\mu^4} \frac{E^4}{R} \bigg/ \frac{q_e^2}{3\epsilon_0 m_e^4} \frac{E^4}{R} = \frac{q_\mu^2}{m_\mu^4} \bigg/ \frac{q_e^2}{m_e^4}. \quad (8)$$

If we note that $q_e = q_\mu$, then we only really care about the mass,

$$= \frac{m_e^4}{m_\mu^4} \approx 5.51 \times 10^{-10} \quad (9)$$

We can see that muons lose about 10^{10} times less energy than electrons do! This is a staggering difference, and a very compelling reason why a muon accelerator is worth considering (even despite some simplifications we made in the calculation). One downside, though, is that the magnetic field (proportional to m) would have to be much stronger. Similarly for the induced emf. Thus, using muons would require more power and higher voltage.

Problem 3



An infinitely long straight wire, shown below, has a radially dependent current density. Let r represent the radial distance in cylindrical coordinates that points away from the axis of the wire $\hat{z}$.

(a) Suppose the volume current density is given by

$$\mathbf{J} = J_0 \left(1 - \frac{r^2}{R^2} \right) \hat{z}, \quad (10)$$

where J_0 is a constant, what is the value of J_0 if the outer radius of the wire is $R = 0.553$ m, and the total current is $I_0 = 158$ A?

The total current I_0 must match with the total integrated current density inside the radius R .

$$\begin{aligned} I_0 &= \int d\mathbf{A} \cdot \mathbf{J} \\ &= J_0 \int_0^{2\pi} d\theta \int_0^R dr \, r \left(1 - \frac{r^2}{R^2} \right) \\ &= 2\pi J_0 \int_0^R dr \left(r - \frac{r^3}{R^2} \right) \\ &= 2\pi J_0 \left(\frac{R^2}{2} - \frac{R^2}{4} \right). \end{aligned} \quad (11)$$

Comparing both sides, you get

$$\begin{aligned} J_0 &= \frac{2I_0}{\pi(R^2)} \\ &= 329 \text{ A/m}^2. \end{aligned} \quad (12)$$

(b) What is the magnitude of the magnetic field at $r_1 = 0.283$ m ($< R$)?

We use Ampère's law. Let us take a circular path C with radius $r_1 = 0.283$ m going in $\hat{\phi}$ direction

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}(r), \quad (13)$$

where

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = 2\pi r_1 B_1, \quad (14)$$

$$\begin{aligned} I_{\text{enc}}(r_1) &= J_0 \int_0^{2\pi} d\theta \int_0^{r_1} dr \, r \left(1 - \frac{r^2}{R^2} \right) \\ &= 71.9 \text{ A}. \end{aligned} \quad (15)$$

Then we get

$$B_1 = 5.08 \times 10^{-5} \text{ T}. \quad (16)$$

(c) What is the magnitude of the magnetic field at $r_2 = 1.10$ m ($> R$)?

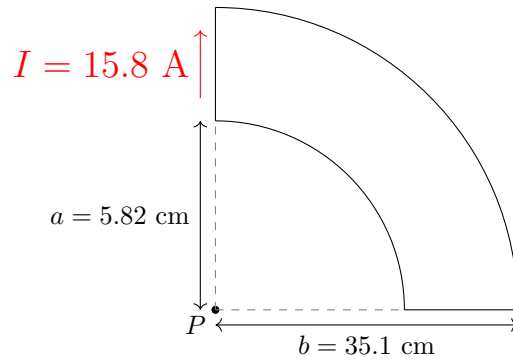
Outside of the wire, the total enclosed current inside of the circular path is I_0 . Then at radius $r_2 = 1.10$ m, the magnetic field is given by

$$\begin{aligned} B_2 &= \frac{\mu_0 I_0}{2\pi r_2} \\ &= 2.87 \times 10^{-5} \text{ T}. \end{aligned} \quad (17)$$

(d) What is the magnitude of the magnetic field at $r = R$?

$$\begin{aligned} B_R &= \frac{\mu_0 I_0}{2\pi R} \\ &= 5.71 \times 10^{-5} \text{ T.} \end{aligned} \tag{18}$$

Problem 4



Consider a magnetic field generated by the current in the 1/4 arc wire shown above.

(a) What is the direction of the magnetic field at point P?

The current components parallel to the radial direction from P does not contribute to the magnetic field at P. Using Biot-Savart law, the magnetic field at point P by the 1/4 arc with radius a is given by

$$\begin{aligned}
 \mathbf{B}_a &= \frac{\mu_0}{4\pi} \int \frac{I d\mathbf{l} \times (-\hat{\mathbf{r}})}{a^2} \\
 &= -\frac{\mu_0}{4\pi} \int_0^{\pi/2} d\phi \frac{Ia\hat{\phi} \times \hat{\mathbf{r}}}{a^2} \quad (\text{In this case, } d\mathbf{l} = (a d\phi) \hat{\phi}) \\
 &= \frac{\mu_0 I}{4\pi a} (\hat{\mathbf{r}} \times \hat{\phi}) \int_0^{\pi/2} d\phi \\
 &= \frac{\mu_0 I}{8a} \hat{\mathbf{z}}.
 \end{aligned} \tag{19}$$

In the same manner, the magnetic field by the outer arc is given by

$$\mathbf{B}_b = -\frac{\mu_0 I}{8b} \hat{\mathbf{z}}. \quad (\text{In this case, } d\mathbf{l} = (-b d\phi) \hat{\phi}) \tag{20}$$

Then the total magnetic field at point P is given by

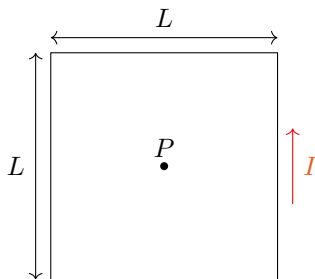
$$\begin{aligned}
 \mathbf{B} &= \mathbf{B}_a + \mathbf{B}_b \\
 &= \frac{\mu_0 I}{8} \left(\frac{1}{a} - \frac{1}{b} \right) \hat{\mathbf{z}}
 \end{aligned} \tag{21}$$

Since $a < b$, the total magnetic field at point P has a direction out of the page (*i.e.* in $\hat{\mathbf{z}}$ direction).

(b) What is the magnitude of the magnetic field at point P?

$$|\mathbf{B}| = \frac{\mu_0 I}{8} \left(\frac{1}{a} - \frac{1}{b} \right) \tag{22}$$

Problem 5



Suppose you have a wire in the shape of a square with side length L and current I going counterclockwise around the square. Our goal in this problem is to calculate the magnetic field at the center of the square.

(a) What can you say about the magnetic fields of each side of the square?

Solution: They are all equal in magnitude since they are equidistant from the center, and equal in direction from the right hand rule;

$$\mathbf{B}(P) = 4\mathbf{B}_{\text{side}}(P) \quad (23)$$

(b) Write down (but don't necessarily solve) an integral for the magnitude of the magnetic field at the center of the square.

Solution: We are going to use Biot-Savart and integrate over the bottom side:

$$\begin{aligned} \vec{B}_{\text{tot}}(P) &= 4\vec{B}_{\text{side}}(P) \\ &= 4 \left(\frac{\mu_0}{4\pi} \int \frac{I d\mathbf{l} \times (-\hat{r})}{r^2} \right) \\ &= -\frac{\mu_0 I}{\pi} \int \frac{d\mathbf{l} \times \vec{r}}{r^3} \end{aligned} \quad (24)$$

We are going to choose to integrate over the bottom side, $y = -L/2$:

$$\begin{aligned} \vec{B}_{\text{tot}}(P) &= -\frac{\mu_0 I}{\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx \times (x\hat{x} + y\hat{y})}{(x^2 + y^2)^{3/2}} \\ &= -\frac{\mu_0 I}{\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx(-\frac{L}{2}\hat{z})}{\left(x^2 + \left(\frac{L}{2}\right)^2\right)^{3/2}} \end{aligned} \quad (25)$$

(c) Solve the integral and find the direction *and* magnitude of the magnetic field at the center of the square. (*Hint: you should use a trig substitution to make your life easier.*)

Solution: From before,

$$B_{\text{tot}}(P) = \frac{\mu_0 I}{\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx(\frac{L}{2}\hat{z})}{\left(x^2 + \left(\frac{L}{2}\right)^2\right)^{3/2}} \quad (26)$$

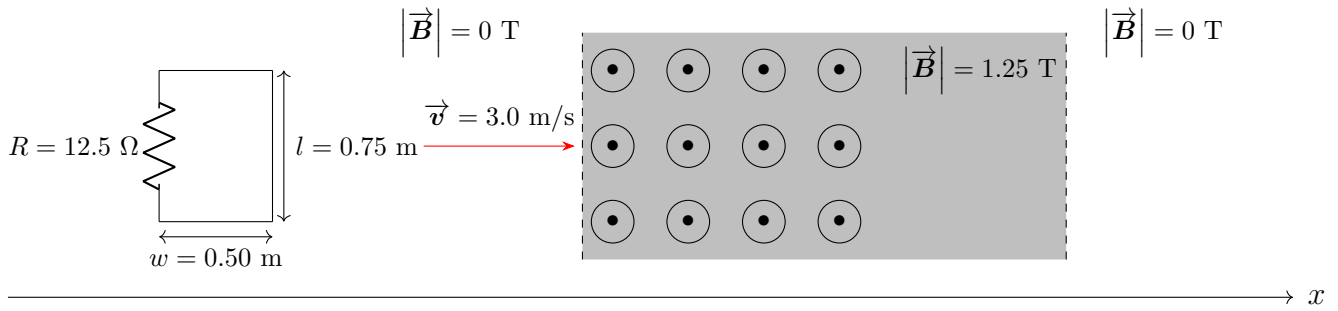
Using the trig sub, $x = \frac{L}{2} \tan(\theta)$, $dx = \frac{L}{2} \sec^2(\theta)d\theta$,

$$\vec{B}_{\text{tot}}(P) = \frac{\mu_0 I}{\pi} \int_{-\pi/4}^{\pi/4} \frac{\left(\frac{L}{2}\right)^2 \sec^2(\theta)d\theta}{\left(\left(\frac{L}{2}\right)^2 (1 + \tan^2(\theta))\right)^{3/2}} \hat{z}. \quad (27)$$

Remembering $\sec^2(\theta) = 1 + \tan^2(\theta)$,

$$\vec{B}_{\text{tot}}(P) = \frac{\mu_0 I}{\pi} \int_{-\pi/4}^{\pi/4} \frac{\left(\frac{L}{2}\right)^2 \sec^2(\theta)d\theta}{\left(\frac{L}{2}\right)^3 \sec^3(\theta)} \hat{z} = \frac{\mu_0 I}{\pi} \int_{-\pi/4}^{\pi/4} \frac{2}{L} \cos(\theta)d\theta \hat{z} = \frac{\mu_0 I}{\pi} \frac{2}{L} \sin(\theta) \Big|_{-\pi/4}^{\pi/4} \hat{z} = \frac{2\sqrt{2}\mu_0 I}{\pi} \hat{z}. \quad (28)$$

Problem 6



(a) Calculate the induced current as the circuit enters the field.

We use Lenz's law or Faraday's law. Since in the field there is a uniform magnetic field, the flux change $d\Phi$ due to the change in the area dA is given by

$$d\Phi = B dA = B l dx. \quad (29)$$

Then the *emf* magnitude is given by

$$\begin{aligned} |\mathcal{E}| &= \left| -\frac{d\Phi}{dt} \right| = |B| \left| \frac{dA}{dt} \right| \\ &= |B| l \left| \frac{dx}{dt} \right| \\ &= |B| l v \\ &= (1.25 \, \text{T}) (0.75 \, \text{m}) (3.0 \, \text{m/s}) \\ &= 2.81 \, \text{V}. \end{aligned} \quad (30)$$

Then the induced current is given by Ohm's law

$$\begin{aligned} I &= \frac{|\mathcal{E}|}{R} \\ &= \frac{2.81 \, \text{V}}{12.5 \, \Omega} \\ &= 0.23 \, \text{A}. \end{aligned} \quad (31)$$

(b) Calculate the induced current when the circuit is totally in the field.

Since the change in area is $dA = 0$, you get

$$|\mathcal{E}| = 0.0 \, \text{V}. \quad (32)$$

Then you get $I = |\mathcal{E}| / R = 0.0 \, \text{A}$.

(c) Calculate the induced current as the circuit moves out of the field.

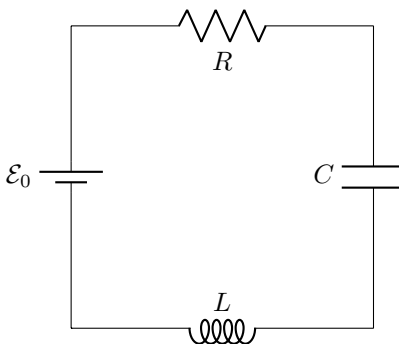
The *emf* magnitude is given by

$$\begin{aligned} |\mathcal{E}| &= \left| -\frac{d\Phi}{dt} \right| = |B| \left| \frac{dA}{dt} \right| \\ &= |B| l \left| -\frac{dx}{dt} \right| \\ &= |B| l v \\ &= (1.25 \, \text{T}) (0.75 \, \text{m}) (3.0 \, \text{m/s}) \\ &= 2.81 \, \text{V}. \end{aligned} \quad (33)$$

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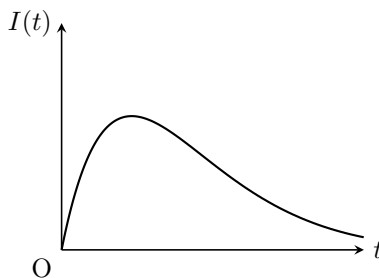
Problem 7



Suppose you have a underdamped RLC circuit connected to a DC source as shown above. At time $t = 0$, the battery is replaced by a closed switch. Find (a) the current of the circuit just when the switch closes and the current when the switch has been closed a long time. (b) Sketch the behavior of the current in the circuit through time. (c) Explain the behavior of the energy of the circuit through time and explain where this energy goes as $t \rightarrow \infty$.

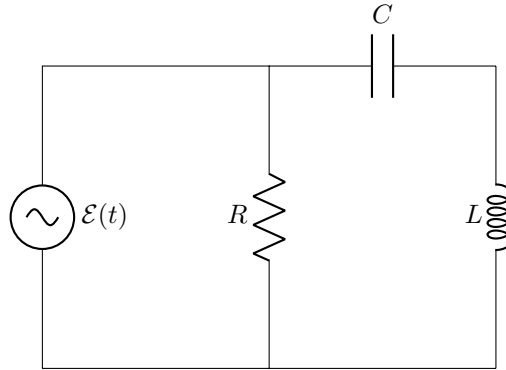
Solution:(a) The current of the circuit just when the switch closes at $t = 0$ is simply $I(0) = 0$ due to the existence of the inductor in the circuit, since the inductor resists the change in the current. The current of the circuit also goes to 0 as $t \rightarrow \infty$ since there is no voltage source to constant supply energy to the circuit to replace the energy dissipated by the the resistor.

(b)



(c) Just like the current, the energy of the circuit will dissipate overtime. This energy will be transferred into heat by the resistor.

Problem 8



Consider a voltage source $\mathcal{E}(t) = \mathcal{E}_0 \cos \omega t$ driving the circuit above, where the positive sense of current is indicated with an arrow. Assuming the inductor's top terminal is its positive reference, find the real-valued amplitude I_L such that the current across the inductor is written

$$I_L(t) = I_L \cos(\omega t + \phi) \quad (35)$$

Find the amplitude, I_L and the phase, ϕ . Find all possible values of the phase ϕ and briefly comment about the conditions for a system with each phase.

Solution: We can find the current through the inductor using Ohm's Law

$$\mathcal{E}(t) = I_L(t) X_{total}$$

$$X_{total} = \sqrt{X_R^2 + (X_L - X_C)^2}$$

The current through the inductor is not the same as the current through the resistor due to the parallel system, therefore the impedance of the resistor can be disregarded. (This can also be explained by a Kirchhoff loop around the outer parts of the circuit).

$$X_{tot} = \sqrt{(\omega L - \frac{1}{\omega C})^2} = \omega L - \frac{1}{\omega C}$$

Solving for I_L , we get

$$I_L(t) = \frac{\mathcal{E}_0 \cos(\omega t + \phi)}{\omega L - \frac{1}{\omega C}}$$

$$I_L = \frac{\mathcal{E}_0}{\omega L - \frac{1}{\omega C}}$$

To find the phase, we use the formula

$$\tan(\phi) = \frac{X_L - X_C}{X_R}$$

We again disregard the resistor to obtain

$$\tan(\phi) = \frac{\omega L - \frac{1}{\omega C}}{0} = \pm \infty$$

with the $\pm$ depending on whether $\omega L > \frac{1}{\omega C}$ or $\omega L < \frac{1}{\omega C}$. Respectively,

$$\phi = \arctan(\pm \infty) = \pm \frac{\pi}{2}$$